### TRANSPORTATION PROBLEM

**Structure**

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### INTRODUCTION

Transportation problem is a special kind of Linear Programming Problem (LPP) in which the objective is to transport goods from a set of sources/origins to a set of destinations in such a manner that the total transportation or shipping cost is minimized. To achieve this objective, we must know about some parameters such as the quantity of available supplies, the quantity demanded and the costs of shipping a unit from various origins to various destinations. Solving LPPs by simplex method discussed in chapter 3 involves a large number of variables and constraints and takes a long time to solve it. So, in this chapter, we shall discuss the methods, which are specifically applied for solving transportation problems. Firstly, we shall explain how to formulate mathematically a transportation problem and methods of finding initial basic feasible solution. After finding the initial basic feasible solution, we discuss how optimality test is

performed by applying the Stepping Stone Method or Modified Distribution Method (MODI)to find whether the obtained feasible solution is optimal or not. In the last, we shall explain the case of unbalanced transportation problems and degeneracy.

### Objectives

After studying this chapter, the reader should be able to:

* + - * Define a transportation problem.
			* Obtain the basic feasible solution of a transportation problem using North – West Corner Rule, Least Cost and Vogel’s Approximation methods.
			* State the conditions for performing optimality test.
			* Explain the algorithm of the Stepping Stone and Modified Distribution (MODI) methods of obtaining the optimal solution of a transportation problem.
			* Solve the transportation problems for special cases such as unbalanced transportation problem, case of degeneracy.

### MATHEMATICAL FORMULATION OF THE TRANSPORTATION PROBLEM

Let there be m origins/ sources of supply 𝑂1, 𝑂2, … , 𝑂𝑖 … 𝑂𝑚 and n destinations𝐷1, 𝐷2, … , 𝐷𝑗 … 𝐷𝑛 . The total number of the capacities of all m origins is assumed to be equal to the total number of the requirements of all n destinations. Let 𝐶𝑖𝑗 be the cost of shipping one unit from origin i to destination j. Let 𝑎𝑖 be the capacity/ availability of items at origin i and 𝑏𝑗, the requirement/demand of the destination

j. Then this transportation problem can be expressed in a tabular form as follows:

|  |  |  |
| --- | --- | --- |
| Origin | Destinations𝐷1 𝐷1 ... 𝐷𝑗 ... 𝐷𝑛 | Availability/ capacity |
| 𝑂1𝑂2𝑂𝑖𝑂𝑚 | 𝐶11 𝐶12 ... 𝐶1𝑗 ... 𝐶1𝑛𝐶21 𝐶22 ... 𝐶2𝑗 ... 𝐶2𝑗. . . .. . . .. . . .𝐶𝑖1 𝐶𝑖2 ... 𝐶𝑖𝑗 ... 𝐶𝑖𝑛. . . .. . . .. . . .𝐶𝑚1 𝐶𝑚2 ... 𝐶𝑚𝑗 ... 𝐶𝑚𝑛 | 𝑎1𝑎2...𝑎𝑖...𝑎𝑚 |
| Requirement/ Demand | 𝑏1 𝑏2 ... 𝑏𝑗 ... 𝑏𝑛 | Total |

The condition for the existence of a feasible solution to a transportation problem is give as

𝑚 𝑛

∑ 𝑎𝑖 = ∑ 𝑏𝑗

𝑖=1 𝑗=1

The above equation tells us that the total requirement/demand equals the total capacity. If it is not so, a dummy origin or destination is created to balance the total capacity and requirement.

Now let 𝑥𝑖𝑗 be the number of units to be transported from origin i to destination j and 𝐶𝑖𝑗 the corresponding

cost of transportation. Then the total transportation cost is ∑𝑚

. ∑𝑛

𝐶𝑖𝑗𝑥𝑖𝑗

Subject to the constraints:

𝑛 𝑛

𝑖=1

𝑛

𝑗=1

∑ 𝑥1𝑗 = 𝑎1 , ∑ 𝑥2𝑗 = 𝑎2 , … , ∑ 𝑥𝑚𝑗 = 𝑎𝑚

𝑗=1

𝑚

𝑗=1

𝑚

𝑗=1

𝑛

……(3)

∑ 𝑥𝑖1 = 𝑏1 , ∑ 𝑥𝑖2 = 𝑏2 , … , ∑ 𝑥𝑖𝑛 = 𝑏𝑛

𝑖=1

𝑖=1

𝑗=1

And 𝑥𝑖𝑗 ≥ 0 for all i = 1, 2, 3... m and j = 1, 2... n.

The Simplex method is regarded as the most generalized method to solve this. However, the solution is very lengthy and takes a long time to solve it since a large number of decision variables and artificial variables are involved. It is far simpler to solve it by transportation method as compared to the Simplex method. In the transportation method, we first obtain the initial basic feasible solution and then perform the optimality test.

**Note:** A transportation problem is said to be balanced if it satisfies the condition ∑𝑚

𝑖=1

𝑎𝑖 = ∑𝑛

𝑏𝑗.

### METHODS OF FINDING INITIAL BASIC FEASIBLE SOUTION

𝑗=1

There are several methods to obtain initial basic feasible solution. Here, we shall discuss the following methods to determine the initial basic feasible solution:

### North-West Corner Rule

1. **Least Cost Method**

### Vogel’s Approximation Method (Penalty or Regret Method)

Vogel’s Approximation method generally gives a solution closer to the optimum solution. Hence, it is preferred to the other two methods.

### North – West Corner Rule

The North – West Corner Rule (NWC)is a simple and efficient method to obtain initial basic feasible solution. Itcan be summarized as follows:

**Step 1:** Start with cell (1, 1) at the north-west corner (upper left-hand corner) of the transportation matrix and allocate as much as possible there.

**Step 2:** Here, we have three cases

1. If the quantity needed at First Destination (𝑏1) is less than the quantity available at First Origin (𝑎𝑖), we allocate a quantity equal to the requirement at First Destination to the cell (1, 1). At

this stage, Column 1 is exhausted, so we cross it out. Since the requirement 𝑏1 is fulfilled, we reduce the availability 𝑎1 𝑏𝑦 𝑏1 and proceed to north-west corner of the resulting matrix, i.e., cell (1,2).

1. If the quantity needed at First Destination (𝑏1) is greater than the quantity available at First Origin(𝑎1), allocate a quantity equal to the quantity available at First Plant/Origin (𝑎1 ) 𝑡𝑜 𝑐𝑒𝑙𝑙 (1, 1).At this stage, Row 1 is exhausted, so we cross it out and proceed to north- west corner of the resulting matrix, i.e., cell (2, 1).
2. If the quantity needed at First Destination (𝑏1) is equal to the quantity available at First Origin (𝑎1), we allocate a quantity equal to the requirement at First Destination (or the quantity available at First Origin). At this stage, both column 1 as well as Row 1 is exhausted. We cross them out and proceed to the north-west corner of the resulting matrix, i.e., cell (2, 2).

**Step 3:** We continue the procedure, until we reach the south – east corner of the original matrix.

* + - 1. **Example.** Find the basic feasible solution of the given transportation problem by applying North

– West Corner rule:

|  |  |  |
| --- | --- | --- |
| WarehouseFactory | D E F G | Capacity |
| ABC | 42 48 38 3740 49 52 5139 38 40 43 | 160150190 |
| Requirement | 80 90 110 220 | 500 |

**Solution.** We start from the North – West corner, i.e., the Factory A and Warehouse D. The quantity needed at the First Warehouse (Warehouse D) is 80, which is less than the quantity available (160) at the First Factory A. Therefore, a quantity equal to the warehouse D is to be allocated to the cell (A, D). Thus, the requirement of Warehouse D is met by Factory A. So, we cross out column 1 and reduce the capacity of Factory A by 80. Then we go to cell (A, E), which is North – West corner of the resulting matrix.

Now, the quantity needed at the second Warehouse (Warehouse E) is 90, which is greater than the quantity available (80) at the First Factory A. Therefore, we allocate a quantity equal to the capacity at Factory A, i.e., 80 to the cell (A, E). The requirement of Warehouse E is reduced to 10. The capacity of Factory A is exhausted and has to be removed from the matrix. Therefore, we cross out row 1 and proceed to cell (B, E).

Now, the quantity needed at the second Warehouse (Warehouse E) is 10, which is less than the quantity available at the Second Factory B, which is 150. Therefore, the quantity 10 equal to the requirement at Warehouse E is allocated to the cell (B, E). Hence, the requirement of Warehouse E is met and we cross out column 1. We reduce the capacity of Factory B by 10 and proceed to cell (B, F).

Again, the quantity needed at the Third Warehouse (Warehouse F) is 110. It is less than the quantity available at the Second Factory (Factory B), which is140. Therefore, a quantity equal to the requirement at Warehouse F is allocated to the cell (B, F). Since the requirement of Warehouse F is met, we cross out Column 1 and reduce the capacity of Factory B by 110. Then we proceed to cell (B, G).Now, the quantity needed at the Fourth Warehouse (Warehouse G) is 220, which is greater than the quantity available at the Second Factory (Factory B).Therefore, we allocate the quantity equal to the capacity of Factory B to the cell (B, G) so that the capacity of Factory B is exhausted and the requirement of Warehouse G is reduced to 190. Hence, we cross out Row 1 and proceed to cell (C, G).

Thus, the allocations given using North – West corner rule are as shown in the following matrix along with the cost per unit of transportation:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| WarehouseFactory | D E F G |  |  | Capacity |
| ABC | 42 80 48 80 38 3740 49 10 52 11 5139 38 40 43 | 30190 |  | 160150190 |
| Requirement | 80 90 110 220 |  |  | 500 |

Thus, the total transportation cost for these allocations

= 42 × 80 + 48 × 80 + 49 × 10 + 52 × 110 + 51 × 30 + 43 × 190

= 3360 + 3840 + 490 + 5720 + 1530 + 8170 =23110

* + - 1. **Exercise.** Find the basic feasible solution of the following problem using North-West Corner Rule:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Origin/Distribution Centre | 1 | 2 | 3 | 4 | 5 | 6 | Availability |
| 1 | 4 | 6 | 9 | 2 | 7 | 8 | 10 |
| 2 | 3 | 5 | 4 | 8 | 10 | 0 | 12 |
| 3 | 2 | 6 | 9 | 8 | 4 | 13 | 4 |
| 4 | 4 | 4 | 5 | 9 | 3 | 6 | 18 |
| 5 | 9 | 8 | 7 | 3 | 2 | 14 | 20 |
| Requirements | 8 | 8 | 16 | 3 | 8 | 21 |  |

**Answer.** Using North - West corner rule, the allocations are to be made as under:

8 units to cells (1,1), 2 units to cell (1,2), 6 units to cell (2,2), 6 units to cell (2,3), 4 units to cell (3 ,3), 6

units to cell (4,3), 3 units to cell (4,4), 8 units to cell (4,5), 1 unit to cell (4,6) and 20 units to cell (5,6) and the transportation cost is equals to 501.

### Least Cost Method

This method is also known as the Matrix Minimum method or Inspection method. It starts by making the first allocation to the cell for which the transportation cost per unit is lowest. The row or column for which the capacity is exhausted or requirement is satisfied is removed from the transportation table. We follow the procedure with the reduced matrix until all the requirements are satisfied. If there is a tie for the lowest cost cell while making any allocation, the choice may be made for a row or a column by which maximum requirement is exhausted. If there is a tie in making this allocation as well, then we can arbitrarily choose a cell for allocation.

The method can be easily explained with the help of the following example.

* + - 1. **Example.** Find the basic feasible solution of the transportation problem of Example 4.3.1.1. by using the Least Cost method.

**Solution.** Here, the least cost is 37 in the cell (A, G). The requirement of the Warehouse G is 220 and the capacity of Factory A is 160. Hence, the maximum number of units that can be allocated to this cell is

160. Thus, Factory A is exhausted. The requirement of Warehouse G is reduced by 160.

Now, the least cost is 38, which is in the cell (C, E). The requirement of the Warehouse E is 90 and the capacity of Factory C is 190. Hence, the maximum number of units that can be allocated to this cell is 90. Moreover, we reduce the capacity of factory C by 90.

The least cost in the matrix is 39, which is in the cell (C, D). The requirement of Warehouse D is 80 and the capacity of Factory C is 100.Hence, the maximum number of units that can be allocated to this cell is

80. The requirement of Warehouse D is exhausted. The capacity of the Factory C is also reduced by 80.

The least cost in this matrix is 40 which is in the cell (C, F). The requirement of Warehouse F is 110 and the capacity of Factory C is 20. Hence, the maximum number of units that can be allocated to this cell is

20. Thus, Factory C is exhausted. The requirement of Warehouse F is reduced by 20. It is now 90 in the reduced matrix.

The least cost is 51 in the cell (B, G) and the requirement of warehouse G is 60 units. So, we allocate 60 units to cell (B, G) and the remaining 90 units to the cell (B, F). Thus, the allocations given using Least Cost method are as shown in the following matrix along with the cost per unit of transportation:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| WarehouseFactory | D |  | E |  | F |  | G | Capacity |
| ABC | 42 |  | 48 |  | 38 |  | 37 | 160 | 160150190 |
|  |
|  | 90 |  |
|  |  |  | 60 |
| 40 |  | 49 |  | 52 |  | 51 |
|  | 39 |  | 90 |  | 20 |  |
| 39 |  | 38 |  | 40 |  | 43 |
| Requirement | 80 |  | 90 |  | 110 |  | 220 | 500 |

Thus, the total transportation cost = 37× 160 + 52 × 90 + 51 × 80 + 38 × 90 + 40 × 20

= 21000

**Note:** This method has reduced the total transportation cost in comparison to the NWC rule.

* + - 1. **Exercise.** Find the basic feasible solution of the following problem using the Least Cost method:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Origin/Distribution Centre | 1 | 2 | 3 | 4 | 5 | 6 | Availability |
| 1 | 4 | 6 | 9 | 2 | 7 | 8 | 10 |
| 2 | 3 | 5 | 4 | 8 | 10 | 0 | 12 |
| 3 | 2 | 6 | 9 | 8 | 4 | 13 | 4 |
| 4 | 4 | 4 | 5 | 9 | 3 | 6 | 18 |
| 5 | 9 | 8 | 7 | 3 | 2 | 14 | 20 |
| Requirements | 8 | 8 | 16 | 3 | 8 | 21 |  |

**Answer.** Using Least Cost method, the allocations are to be made as under:

4 units to cell (1,1), 3 units to cell (1,4), 3 units to cell (1,6), 12 units to cell (2,6), 4 units to cell (3,1), 8

units to cell (4,2), 10 units to cell (4,3), 6 units to cell (5,3), 8 units to cell (5,5) and 6 units to cell (5,6). The transportation cost = 278.

### Vogel’s Approximation Method (VAM)

We describe the step by step procedure for finding the initial basic feasible solution by Vogel’s Approximation method (Penalty method) in the following steps:

1. In the transportation table calculate penalties for each row (column), by taking the difference between the least and second least costs in the same row (column). We display it to the right (below) of that row (column) in a new column (row) formed by extending the table on the right (bottom). The new column and row formed by extending the table at the right and bottom are labelled as penalty column and penalty row, respectively. The differences noted in the penalty row or penalty column indicates the penalty or extra cost. If two cells in a row (or column) contain the same least costs then the difference is taken as zero.
2. Select the row or column with the largest penalty (largest difference) and allocate the maximum possible units to the least cost cell in the selected column or row. If there is a tie in the values of penalties, the choice may be made for that row or column, which has the least cost. In case there is a tie in such least cost as well, choice may be made from that there is a tie in such least cost as well, choice may be made from that row or column by which maximum requirements are exhausted

.The cell so chosen is allocated the units and the corresponding exhausted row or column is removed or ignored from further consideration.

1. Now, we determine the column and row differences for the reduced transportation table and repeat the procedure until all column and row totals are exhausted.

This method is also known as the penalty method**.** Let us understand the procedure with the help of an example.

* + - 1. **Example.** Apply the Vogel’s Approximation Method for finding the Basic Feasible Solution for the transportation problem of Example 4.3.1.1.

**Solution.** In the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write 1 in a new column created on the right. It is labelled Penalty.Similarly, the differences between the least and the second least costs in the second and third row, respectively, are 49 – 40 = 9 and 39 – 38 = 1. So, we write the values (differences), i.e., 9 and 1 in the penalty column.

Next, we find the differences of the least and second least elements of each of the columns D, E, F and G. These are 40-39 = 1, 48-38 = 10, 40-38 = 2 and 43-37 = 6, respectively. We write them in a newly created penalty row at the bottom of the table.

We now select the largest of these differences in the penalty row and column, which are 10 in this case. This value (10) corresponds to the second column (Column E) and the least cost in the column is 38. Hence the allocation of 90 units (the maximum requirement of warehouse E) is to be made in the cell (C, E) from Factory C. Since the column corresponding to E is exhausted, it is removed for the next reduced matrix and the capacity of C is reduced by 90.

We now take the differences between the least and the Second least cost for each row and column of the reduced matrix. In the first row, the least and the second least costs are 37 and 38 and their differences is

1. We write it in the newly created penalty column. Similarly, we write the second difference element 51- 40 = 11 and third difference element 40-39 = 1 in the second and third row of this column. Likewise, the differences of the smallest and second smallest elements of each of the columns D, F, and G are 40–39 = 1, 4–38 = 2 and 43–37 = 6, respectively. We write these in a newly created penalty row at the bottom of the table.

Now, we select the largest of these differences in the penalty row and column, which is 11 in this case. This value (11) corresponds to Row B. Since the least cost in row is 40, we allocate 80 units (the maximum requirement of Warehouse D) to the cell (B, D). Thus, the requirement of Warehouse D is exhausted and we can remove it. We also reduce the capacity of Factory B by 80 in the next reduced matrix.

Again, in the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write it to the right of this row in the newly created penalty column. Similarly, the second and third elements in the second and third rows of this column are 52–51=1 and

43–40 = 3, respectively. We write these in a newly created penalty row at the bottom of the table. Now, we select the largest of these differences, which are 6 in this case. It corresponds to Column G and the least cost in this column is 37. Hence, we allocate 160 units (the maximum capacity of Factory A) to the cell (A, G). Since Row A is exhausted, it is removed for the next reduced matrix. We also reduce the requirement of Warehouse G by 160 units.

Once again, the difference of the least costs in the first row is 52 – 51 = 1. We write it in the newly created penalty column. Similarly, for the second row, the difference is 43-40 = 3. Likewise, the differences of the least and second least elements of each of the columns F and G are 52 – 40 = 12 and 51 – 43 = 8, respectively. We write them in the newly created penalty row at the bottom of the table. The largest of these differences is 12 in this case. It corresponds to Column F and the least cost in this column is 40. Hence, we allocate 100 units from the row (the maximum capacity of Factory C) to the cell (C, F). Since Row C is exhausted, it is removed and the requirement of Warehouse F is reduced to 10 for the next reduced matrix.

At the end, of the 70 units available in Factory B, we allocate 60 units to the lower cost (51), i.e., to the cell (B, G) and the remaining 10 units to the cell (B, F).

The entire procedure of allocating units by Vogel’s Approximation Method is given in the following table:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| WarehouseFactory | D |  | E F G | Capacity | Diff1 | Diff2 | Diff3 | Diff4 |
| A | 42 |  | 48 38 **37 160** | 160 | 1 | 1 | 1 | - |
| B | **40** | **80** | 49 **52 1051 60** | 150 | 9 | 11\* | 1 | 1 |
| C | 39 |  | **38 90 40 100** 43 | 190 | 1 | 1 | 3 | 3 |
| Requirement | 80 |  | 90 110 220 | 500 |  |  |  |  |
| Diff1 | 1 |  | 10\* 2 6 |  |
| Diff2 | 1 |  | - 2 6 |
| Diff3 | - |  | - 2 6\* |
| Diff4 | - |  | - 12\* 8 |

Thus, the total transportation cost

=40 × 80 + 38 × 90 + 52 × 10 + 40 × 100 + 37 × 160 + 51 × 60

= 3200 + 3420 + 520 + 4000 + 5920 + 306 = 20120

**Note:** The total transportation cost obtained above is the lowest total transportation cost among the three methods. Clearly the solution obtained by VAM is nearest to the optimal solution.

* + - 1. **Exercise.** Find the basic feasible solution of the following problem using Vogel’s Approximation method:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Origin /Distribution Centre | 1 | 2 | 3 | 4 | 5 | 6 | Availability |
| 1 | 4 | 6 | 9 | 2 | 7 | 8 | 10 |
| 2 | 3 | 5 | 4 | 8 | 10 | 0 | 12 |
| 3 | 2 | 6 | 9 | 8 | 4 | 13 | 4 |
| 4 | 4 | 4 | 5 | 9 | 3 | 6 | 18 |
| 5 | 9 | 8 | 7 | 3 | 2 | 14 | 20 |
| Requirement | 8 | 8 | 16 | 3 | 8 | 21 |  |

**Answer.** The total transportation cost= 242.

### ASSIGNMENT PROBLEMS

**Structure**

* 1. Introduction
	2. Assignment Problems
	3. Hungarian Method
	4. Unbalanced Assignment Problem
	5. Case of Maximization of an Assignment Problem
	6. Travelling Salesman Problem
	7. Check Your Progress
	8. Summary

### INTRODUCTION

Assignment problem is a special type/case of transportation problem and hence is of linear programming problem which deals with the allocation of the various resources to the various activities on one to one basis. It does it in such a manner that the cost or time involved in the process is minimum and profit or sale is maximum. In chapter 4, we have already discussed methods of finding optimal solution for the given problem but here we discuss another method namely Hungarian method for solving an assignment problem. Hungarian method is shorter and easier than stepping stone and MODI methods which we have discussed in previous chapter. In this chapter, we shall explain the assignment problems including travelling salesman problem and apply Hungarian method for solving these problems.

### Objectives

After studying this chapter, reader should be able to explain the following concepts like:

* Assignment Problems
* Hungarian Method.
* Unbalanced Assignment Problem
* Case of Maximization of an Assignment Problem
* Travelling Salesman Problem

### ASSIGNMENT PROBLEMS

An assignment problem may be considered as a special type of transportation problem in which there are as many jobs/sources as the number of machines/destinations so that the jobs can be assigned to machines in a one-to-one way only. The capacity of each source as well as the requirement of each destination is taken as 1. The main difference between an assignment problem and transportation problem is that in the case of assignment problem, the given matrix must necessarily be a square matrix which is not the condition for a transportation problem.

Let there be n persons and n jobs and let 𝐶𝑖𝑗 represents the amount of time taken by 𝑖𝑡ℎ person to complete the 𝑗𝑡ℎ job then our objective is assignment of jobs on one-to-one basis in such a way that the total cost is minimum. The assignment problem can be stated in the form of an n×n matrix of real numbers called the cost matrix as given in the following table:

|  |  |
| --- | --- |
| **Person** | **Job****1 2 … j … n** |
| 12...i...n | 𝐶11𝐶12 … 𝐶1𝑗 … 𝐶1𝑛𝐶21𝐶22 … 𝐶2𝑗… 𝐶2𝑛...𝐶𝑖1𝐶𝑖2… 𝐶𝑖𝑗 … 𝐶𝑖𝑛...𝐶𝑛1𝐶𝑛2… 𝐶𝑛𝑗… 𝐶𝑛𝑛 |

Let 𝑥𝑖𝑗 denote the 𝑗𝑡ℎ job assigned to the 𝑖𝑡ℎ person. Then, mathematically, the assignment problem can be stated as follows:

Minimize Z = ∑𝑛

𝑖=1

𝑛

𝑗=1

∑

𝑐𝑖𝑗𝑥𝑖𝑗 , where i= 1, 2, …, n and j=1, 2, …, n.

subject to

1 𝑖𝑓 𝑡ℎ𝑒 𝑖𝑡ℎ person is assigned the 𝑗𝑡ℎ job

𝑥𝑖𝑗 = {

0 𝑖𝑓 𝑡ℎ𝑒 𝑖𝑡ℎ person is not assigned the 𝑗𝑡ℎ job

Since one job assigned to one person only, we have

𝑥𝑖1 + 𝑥𝑖2+ … + 𝑥𝑖𝑛 = 1, i= 1, 2, …, n

𝑥1𝑗 + 𝑥2𝑗+ … + 𝑥𝑛𝑗 = 1, j= 1, 2, …, n

**Note:** The constant 𝑐𝑖𝑗 in the above problem represents time, in many situations it may be cost or some other parameter which is to be minimized in the assignment problem under consideration.

**Remark**: An assignment problem can be looked upon as a special type of transportation problem in which the jobs stands for sources, the machines for destinations and all the availabilities and requirements are equal to one.

### HUNGARIAN METHOD

Hungarian method is also known as Reduced Matrix Method, it is an efficient method for solving assignment problems. Hungarian method is developed by Hungarian mathematician D. Konig. The step by step procedure for obtaining an optimal solution of an assignment problem are as follows:

1. Develop the cost table from the given problem then check whether the given matrix is square i.e. is number of sources/machines is equal to the number of destinations/jobs. If not, make it square by adding a suitable number of dummy row (or column) with 0 cost/time element.
2. Locate the smallest cost element in each row of the given cost matrix and then subtract the smallest element of each column from every element of that column.
3. In the resulting cost matrix, locate the smallest element in each column and subtract the smallest element of each column from every element of that column.
4. In the modified matrix, search for an optimal assignment as follows:
5. Examine the row successively until a row with exactly one zero is found. Draw a rectangle around this zero like 0 and cross out all other zeroes in the corresponding column. Proceed in this way until all the row have been considered. If there is more than one zero in any row, don’t touch that row, pass on to the next row.
6. Repeat step (a) above for the columns of the resulting cost matrix.
7. If a row or column of the reduced matrix contains more than one zeroes, arbitrarily choose a row or column having the minimum number of zeroes. Arbitrarily select any zero in the row or column so chosen. Draw a rectangle around it and cross out all the zeroes in the corresponding row and column. Repeat steps (a), (b), and (c) until all the zeroes have either been assigned (by drawing a rectangle around them) or crossed.
8. If each row or column of the resulting matrix has one and only one assigned zero, i.e. number of assigned zeroes are equal to the number of rows/columns, then the optimum assignment is made in the cells corresponding to 0 and the optimum solution of the problem is attained and we can stop here.

Otherwise, go to the next step.

1. Draw the minimum number of horizontal and/or vertical lines through all the zeroes as follows:
	1. Mark (√ ) the rows in which assignment has not been made.
	2. Mark (√ ) column, that have zeroes in the marked rows.
	3. Mark (√ ) rows( not already marked ) which have assignments in marked columns. Then mark (√ ) columns, which have zeroes in newly marked rows, if any. Mark (√ ) rows (not already marked ), which have assignments in these newly marked columns.
2. Revise the cost matrix as follows:
	1. Find the smallest elements not covered by any of the lines.
	2. Subtract this from all the uncovered elements and add it to the elements at the intersection of the two lines.
	3. Other elements covered by the lines remain unchanged.
3. Repeat the procedure until an optimal solution attained.

**Note:** By drawing lines through all the unmarked rows and marked columns, we will get the required minimum number of lines.

The following example illustrate the method.

* + 1. **Example.** A computer centre has four expert programmers & needs to develop four application programmes. The head of the computer Centre, estimates the computer time (in minutes) required by the respective experts to develop the application programmes as follows:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **PROGRAMMES****PROGRAMMERS** | **A** | **B** | **C** | **D** |
| **1** | 120 | 100 | 80 | 90 |
| **2** | 80 | 90 | 110 | 70 |
| **3** | 110 | 140 | 120 | 100 |
| **4** | 90 | 90 | 80 | 90 |

Find the assignment pattern that minimize the time required to develop the application programmes.

**Solution.** Let us subtract the minimum element of each row from every element of that row. Note that the minimum element in the first row is 80. So, 80 is subtracted from every element of the first row. Similarly, we obtain the elements of the other rows of the resulting matrix. Thus, the modified matrix is:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **A** | **B** | **C** | **D** |
| **1** | 40 | 20 | 0 | 10 |
| **2** | 10 | 20 | 40 | 0 |
| **3** | 10 | 40 | 20 | 0 |
| **4** | 10 | 10 | 0 | 10 |

Let us now subtract the minimum element of each column from every element of that column in the resulting matrix. The minimum element in the first column is 10. So, 10 is to be subtracted from every element of the first column. Similarly, we obtain the elements of the other columns of the resulting matrix. Thus, the resulting matrix is:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **A** | **B** | **C** | **D** |
| **1** | 30 | 10 | 0 | 10 |
| **2** | 0 | 10 | 40 | 0 |
| **3** | 0 | 30 | 20 | 0 |
| **4** | 0 | 0 | 0 | 10 |

Now starting from the first row onward, we draw a rectangle around the zero in each row having a single zero and cross all other zeros in the corresponding column. Here, in the very first row we find a single zero. So, we draw a rectangle around it and cross all the other zeroes in the corresponding column. We get

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **A** | **B** | **C** | **D** |
| **1** | 30 | 10 |  | 0 |  | 10 |
| **2** | 0 | 10 | 40 | 0 |
| **3** | 0 | 30 | 20 | 0 |
| **4** | 0 | 0 | 0 | 10 |

In the second, third and fourth row, there is no single zero. Hence, we move column – wise. In the second column, we have a single zero. Hence, we draw a rectangle around it and cross all other zeroes in the corresponding row. We get

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **A** | **B** | **C** | **D** |
| **1** | 30 | 10 |  | 0 |  | 10 |
| **2** | 0 | 10 | 40 | 0 |
| **3** | 0 | 30 | 20 | 0 |
| **4** | 0 |  | 0 |  | 0 | 10 |

In the matrix above, there is no row or column, which has a single zero. Therefore, we first move row – wise to locate the row having more than one zero. The second row has two zeroes. So, we draw a rectangle arbitrarily around one of these zeroes and cross the other one. Let us draw a rectangle around the zero in the cell (2, A) and cross the zero in cell (2, D). We cross out the other zeroes in the first column. Note that we could just as well have selected zero in the cell (2, D), drawn a rectangle around it and crossed all other zeroes. This would have led to an alternative solution.

In this way, we are left with only one zero in every row and column around which a rectangle has been drawn. This means that we have assigned only one operation to one operator. Thus, we get the optimum solution as follows:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **A** | **B** | **C** | **D** |
| **1** | 30 | 10 |  | 0 |  | 10 |
| **2** |  | 0 |  | 10 | 40 | 0 |
| **3** | 0 | 30 | 20 |  | 0 |  |
| **4** | 0 |  | 0 |  | 0 | 10 |

Note that the assignment of jobs should be made on the basis of the cells corresponding to the zeroes around which rectangles have been drawn. Therefore, the optimum solution for this problem is:

1 → C, 2 → A, 3 → D, 4 → B

This means that programmer 1 is assigned programme C, programmer 2 is assigned programme A, and so on. The minimum time taken in developing the programmes is

= 80 + 80 + 100 + 90 = 350 min.

* + 1. **Example.** A company is producing a single product and selling it through five agencies situated in the different cities. All of a sudden, there is a demand for the product in five more cities that do not have any agency of the company. The company is faced with the problem of deciding on how to assign the existing agencies to dispatch the product to the additional cities in such a way that the travelling distance is minimized. The distances (in km) between the surplus and the deficit cities are given in the following distance matrix:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Deficit citySurplus city | I | II | III | IV | V |
| A | 160 | 130 | 175 | 190 | 200 |
| B | 135 | 120 | 130 | 160 | 175 |
| C | 140 | 110 | 155 | 170 | 185 |
| D | 50 | 50 | 80 | 80 | 110 |
| E | 55 | 35 | 70 | 80 | 105 |

Determine the optimum assignment schedule.

**Solution:** Subtracting the minimum element of each row from every element of that row and then subtracting the minimum element of each column from every element of that column, we have

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **I** | **II** | **III** | **IV** | **V** |
| **A** | 30 | 0 | 35 | 30 | 15 |
| **B** | 15 | 0 | 0 | 10 | 0 |
| **C** | 30 | 0 | 35 | 30 | 20 |
| **D** | 0 | 0 | 20 | 0 | 5 |
| **E** | 20 | 0 | 25 | 15 | 15 |

We now assign zeroes by drawing rectangles around them as explained in above example. Thus, we get

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **I** | **II** | **III** | **IV** | **V** |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **A** | 30 |  | 0 |  | 35 | 30 | 15 |
|  |
| **B** | 15 | 0 |  | 0 |  | 10 | 0 |
| **C** | 30 | 0 | 35 | 30 | 20 |
| **D** |  | 0 |  | 0 | 20 | 0 | 5 |
| **E** | 20 | 0 | 25 | 15 | 15 |

Since the number of assignments is less than number of rows (or columns), we proceed from step 5 onwards of the Hungarian method described as follows:

* + - 1. We mark (√ ) the rows in which the assignment has not been made . These are the 3rd& 5th row.
			2. We mark (√ ) the columns which have zeroes in the marked rows . This is the 2nd column.
			3. We mark (√ ) the rows which have assignments in marked columns . This is the 1st row.
			4. Again we mark (√ ) the columns which have zeroes in the newly marked row . This is the 2nd column (which has been already marked).

There is no other such column. So, we have

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **I** | **II** | **III** | **IV** | **V** |  |
| **A** | 30 |  | 0 |  | 35 | 30 | 15 | √ |
| **B** | 15 | 0 |  | 0 |  | 10 | 0 |  |
| **C** | 30 | 0 | 35 | 30 | 20 | √ |
| **D** |  | 0 |  | 0 | 20 | 0 | 5 |  |
| **E** | 20 | 0 | 25 | 15 | 15 | √ |
|  |  | **√** |  |  |  |  |

We draw straight lines through unmarked rows and marked columns as follows:

√

15

30

35

**V**

**IV**

**III**

**II**

**I**

0

0

20

0

5

**√**

√

15

15

25

0

20

**E**

**D**

√

20

30

35

0

30

**C**

0

10

0

0

15

**B**

0

30

**A**

We proceed as follows, as explained in the step 6 of the Hungarian method:

1. We find the smallest element in the matrix not covered by any of the lines. It is 15 in this case.
2. We subtract the number ‘15’ from all uncovered elements and add it to the elements at the intersection of the two lines.
3. Other elements covered by the lines remain unchanged. Thus, we have

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **I** | **II** | **III** | **IV** | **V** |
| **A** | 15 | 0 | 20 | 15 | 0 |
| **B** | 15 | 15 | 0 | 10 | 0 |
| **C** | 15 | 0 | 20 | 15 | 5 |
| **D** | 0 | 15 | 20 | 0 | 5 |
| **E** | 5 | 0 | 10 | 0 | 0 |

We repeat steps 1 to 4 of the Hungarian method and obtain the following matrix:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **I** | **II** | **III** | **IV** | **V** |
| **A** | 15 | 0 | 20 | 15 |  | 0 |  |
| **B** | 15 | 15 |  | 0 |  | 10 | 0 |
| **C** | 15 |  | 0 |  | 20 | 15 | 5 |
| **D** |  | 0 |  | 15 | 20 | 0 | 5 |
| **E** | 5 | 0 | 10 |  | 0 |  | 0 |

Since each row and each column of this matrix has one and only one assigned 0, we obtain the optimum assignment schedule as follows:

A → V, B → III, C → II, D → I, E→ IV

Thus, the minimum distance is 200+130+110+50+80 = 570 km.

* + 1. **Exercise.** A solicitor’s firm employs typists on an hourly piece – rate basis for their daily work. There are five typists for service and their charges and speeds are different. According to the contract, only one job is given to one typist. Find the least cost allocation for the following data:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **P** | **Q** | **R** | **S** | **T** |
| **A** | 85 | 75 | 65 | 85 | 75 |
| **B** | 90 | 180 | 66 | 90 | 78 |
| **C** | 75 | 66 | 57 | 75 | 69 |
| **D** | 80 | 72 | 60 | 80 | 72 |
| **E** | 76 | 64 | 56 | 72 | 68 |

**Answer.** Applying the Hungarian method of solving an assignment problem, we finally get

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | P | Q | R | S | T |
| A | 2 | 4 | 2 | 4 | 0 |
| B | 4 | 106 | 0 | 6 | 0 |
| C | 0 | 0 | 2 | 2 | 2 |
| D | 0 | 4 | 0 | 2 | 0 |
| E | 2 | 2 | 2 | 0 | 2 |

Thus, the least cost allocation is given by:

A->T, B->R, C->Q, D->P, E->S

and the total minimum cost is Rs. (75+66+66+80+72)

=Rs.359.

### UNBALANCED ASSIGNMENT PROBLEM

Some assignment problems may be unbalanced, i.e. the number of machines may be different from the number of jobs. In this case, in the obtained matrix the number of rows is not equal to the number of columns and the problem said to be an unbalanced Assignment problem. Such a problem is handled by introducing dummy row(s) if the number of rows is less than the number of columns and dummy column(s) if the number of columns is less than the number of rows. All the elements of such a dummy row/column are taken as zero. After creating dummy rows or columns, we get a balanced assignment problem and now we solved it by Hungarian method.

The following example will make the procedure clear.

* + 1. **Example.** To stimulate interest and provide an atmosphere for intellectual discussion, the faculty of mathematical sciences in an institute decides to hold special seminars four contemporary topics - Statistics, Operations Research, Discrete Mathematics, Matrices. Each such seminar is to be held once a week. However, scheduling these seminars (one for each topic and not more than one seminar per a day) has to be done carefully so that the numbers of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Statistics** | **Operations Research** | **Discrete mathematics** | **Matrices** |
| **Monday** | 50 | 40 | 60 | 20 |
| **Tuesday** | 40 | 30 | 40 | 30 |
| **Wednesday** | 60 | 20 | 30 | 20 |
| **Thursday** | 30 | 30 | 20 | 30 |
| **Friday** | 10 | 20 | 10 | 30 |

Find an optimal schedule for the seminars. Also fine the number of students who will be missing at least one seminar.

**Solution.** Here the number of rows is 5 and the number of columns is 4. Therefore, the given assignment problem is unbalanced. As the number of columns is one less than the number of rows, we introduce one dummy column to convert the given assignment problem into a balanced problem. The number of students in each cell of this column is taken as zero. Thus, the problem takes the following form:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Statistics | Operations Research | Discrete mathematics | Matrices | Dummy |
| Monday | 50 | 40 | 60 | 20 | 0 |
| Tuesday | 40 | 30 | 40 | 30 | 0 |
| Wednesday | 60 | 20 | 30 | 20 | 0 |
| Thursday | 30 | 30 | 20 | 30 | 0 |
| Friday | 10 | 20 | 10 | 30 | 0 |

Now, on applying the Hungarian method (Steps 1 to 4), we get

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Statistics | Operations Research | Discrete mathematics | Matrices | Dummy |
| Monday | 40 | 20 | 50 |  | 0 |  | 0 |
| Tuesday | 30 | 10 | 30 | 10 |  | 0 |  |
| Wednesday | 50 |  | 0 |  | 20 | 0 | 0 |
| Thursday | 20 | 10 | 10 | 10 | 0 |
| Friday |  | 0 |  | 0 | 0 | 10 | 0 |

Since the number of assigned zeroes < number of rows, we apply Step 5 of the Hungarian method and draw the minimum number of horizontal/ vertical lines that cover all the zeros as shown in the following table:

40

20

50

0

50

0

20

0

0

0

0

10

0

0

Friday

0

10

10

10

20

Thursday

Wednesday

0

10

30

10

30

Tuesday

0

Monday

Dummy

Matrices

Discrete mathematics

Operations Research

Statistics

We select the minimum element from amongst the uncovered elements, which in 10 in this case. We subtract this element, i.e., 10 from each uncovered element and add it to the elements which lie at the intersection of the horizontal/vertical lines. Other covered elements will remain unaltered. Thus, the resulting matrix is:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 40 | 20 | 50 | 0 | 10 |
| 20 | 0 | 20 | 0 | 0 |
| 50 | 0 | 20 | 0 | 10 |
| 10 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 10 | 10 |

Now on applying the Hungarian method, we have

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 40 | 20 | 50 | 0 | 10 |
| 20 | 0 | 20 | 0 |  | 0 |  |
| 50 |  | 0 |  | 20 | 0 | 10 |
| 10 | 0 |  | 0 |  | 0 | 0 |
|  | 0 |  | 0 | 0 | 10 | 10 |

Since each row and each column of the matrix has one and only one assigned 0, optimum assignment is made in the cells containing those zeroes around which rectangles have been drawn as Monday → Matrices, Wednesday → Operations Research, Thursday → Discrete Mathematics, Friday → Statistics The total number of students who will be missing at least one seminar = 20 + 20 + 20 + 10 = 70

### CASE OF MAXIMIZATION OF AN ASSIGNMENT PROBLEM

All problems dealt with so far were all cost-minimizing problems but assignment problem exists with profit maximization problem also. For example, profits (or anything else like revenues), which need maximization may be given in the cells instead of costs/ times. The method of solving such problems is a simple modification of the method of solving cost-minimizing assignment problems. To solve such a problem, we find the opportunity loss matrix by subtracting the value of each cell from the largest value chosen from amongst all the given cells. When the value of a cell is subtracted from the highest value, it gives the loss of amount caused by not getting the opportunity which would have given the highest value. The matrix so obtained is handled in the same way as the minimization problem. The following example illustrate the method.

* + 1. **Example.** Five salesmen are to be assigned to five districts. Estimates of sales revenue (in thousands) for each salesman are given as following:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **A** | **B** | **C** | **D** | **E** |
| **1** | 32 | 38 | 40 | 28 | 40 |
| **2** | 40 | 24 | 28 | 21 | 36 |
| **3** | 41 | 27 | 33 | 30 | 27 |
| **4** | 22 | 38 | 41 | 36 | 36 |
| **5** | 29 | 33 | 40 | 35 | 39 |

Find the assignment pattern that maximizes the sales revenue.

**Solution.** Since we are to maximize the sales revenue, we need to convert it into minimization form before applying the Hungarian method. For this, we obtain the opportunity loss matrix by subtracting every element in the given table from the largest element is 41. Thus, we obtain the following opportunity loss matrix:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 9 | 3 | 1 | 13 | 1 |
| 1 | 17 | 13 | 20 | 5 |
| 0 | 14 | 8 | 11 | 4 |
| 19 | 3 | 0 | 5 | 5 |
| 12 | 8 | 1 | 6 | 2 |

Now, we apply the Hungarian method (Steps 1 to 4) and finally obtain the following result matrix:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | A | B | C | D | E |
| 1 | 8 |  | 0 |  | 0 | 7 | 0 |
| 2 |  | 0 |  | 14 | 12 | 14 | 4 |
| 3 | 0 | 12 | 8 | 6 | 4 |
| 4 | 19 | 1 |  | 0 |  | 0 | 5 |
| 5 | 11 | 5 | 0 |  | 0 |  | 1 |

Since the number of assigned zero is less than the number of rows, we apply Step 5 of the Hungarian method and draw the minimum number of horizontal/vertical lines that cover all the zeroes as shown in the following table:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | A | B | C | D | E |  |
| 1 |  | 8 |  |  |  |  | 0 | 7 | 0 |  |
|  |  |  | 0 |  |  |  |  |  |
| 2 |  |  | 0 |  | 14 | 12 | 14 | 4 | √ |
| 3 | 0 |  | 12 | 8 | 6 | 4 | **√** |
| 4 |  | 19 |  | 1 |  |  |  | 0 | 5 |  |
|  |  |  |  | 0 |  |  |  |  |
| 5 |  | 11 |  | 5 | 0 |  | 0 |  | 1 |  |
|  |  |  |  |  |  |  |  |  |
|  | √ |  |  |  |  |  |

Let us now, select the minimum element from amongst the uncovered elements, which is 4 in the case. We subtract the element 4 from each of the uncovered elements and add it to the elements which lie at the intersection of the horizontal and vertical lines. Other covered elements remain unaltered. Then applying the Hungarian method to the resulting matrix. We get

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | A | B | C | D | E |
| 1 | 12 |  | 0 |  | 0 | 7 | 0 |
| 2 |  | 0 |  | 10 | 8 | 10 | 0 |
| 3 | 0 | 8 | 4 | 2 |  | 0 |  |
| 4 | 23 | 1 |  | 0 |  | 0 | 5 |
| 5 | 15 | 5 | 0 |  | 0 |  | 1 |

Since the number of assigned zeroes is equal to the number of rows, the optimum assignment has been attained and is given as

1 → B, 2 → A, 3 → E, 4 → C, 5 → D

Thus, the maximum sales revenue = 38+ 40 +37 +41 +35 thousand rupees

= 191 thousand rupees.